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Similarity solutions and wave propagation in a reactive polytropic gas

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Abstract. We consider similarity solutions of the ZND model for detonation waves. Assuming as boundary condition the RH relations for the precursor shock, we obtain exact similarity solutions corresponding to reaction rates compatible with the associated stretching group of transformations which leaves invariant the governing system. The location of weak discontinuities across a similarity line and their evolution laws are determined.

1. Introduction

In this paper we consider a binary reacting mixture produced by a plane shock wave (the precursor shock) which occurs when an infinite piston, moving at constant velocity u_p, shocks a polytropic reactant gas at rest. We assume at first the shock to be purely fluid-dynamic. When the gas particles move across this shock the reaction begins [1]. Behind the precursor shock, in the reacting shocked gas, the usual conservation laws must be supplemented by a chemical reaction equation. The reaction propagates together with the shock at the same detonation velocity D [2].

The governing equations, introduced in Section 2, are given by the ZND (Zel'dovic, von Neumann, Doering) model [2, 3] that consists of the one-dimensional, adiabatic, inviscid-fluid equations to which we must add the progress equation of the chemical reaction and the constitutive relations for the internal energy and reaction rate. Moreover, after having written the equations in Lagrangian coordinates, we carry out a similarity analysis according to invariance-group methods [4, 5] used in [3]. In Section 3, by using the associated-group concept [6], we consider the possibility to obtain exact solutions [7, 8] by assuming the RH relations as boundary conditions. Finally, in Section 4, by following the procedure suggested in [7], we determine the location of weak discontinuities of a similarity solution across a similarity line in the x, t plane, as well as their evolution laws [9].

2. Governing equations and similarity analysis

In writing the governing equations, as is often convenient in computational work, we replace the Eulerian coordinate x by a Lagrangian coordinate defined by

$$\mathrm{d}h = \frac{v}{v_0}(\mathrm{d}x - u\,\mathrm{d}t), \tag{2.1}$$

where the quantities v, u, v_0 denote, respectively, the specific volume of mixture, the particle velocity and the specific volume of reactant. Moreover, p will denote the pressure, and the dimensionless quantity λ (mass fraction of product) is used as a progress variable for an irreversible binary chemical reaction. It is assumed that both the reacting species satisfy the polytropic gas equation of state with the same constant γ. Denoting by e the internal energy of the mixture and taking into account the well-known assumptions [1, 2] about the state equations for reacting mixtures, we can write

$$e = \frac{pv}{\gamma - 1} - \lambda q, \tag{2.2}$$

where q is the energy released per unit of mass in the chemical reaction.

As the flow is supposed to be adiabatic, by taking account of (2.2), the thermodynamic relation

$$\mathrm{d}e = p\mathrm{d}v, \tag{2.3}$$

gives the following equation for the conservation of energy:

$$p_t + \frac{\gamma p}{v} v_t - \frac{\gamma - 1}{v} q\lambda_t = 0. \tag{2.4}$$

Moreover, the equation of reaction progress is given by

$$\lambda_t = Q(v, p, \lambda) \tag{2.5}$$

where Q is the reaction rate, whose value ahead of the shock is zero.

By coupling the usual equations for the conservation of mass and momentum with equation (2.4), where we take into account (2.5), we obtain the following system:

$$\begin{aligned} &v_t - v_0 u_h = 0, \quad u_t + v_o p_h = 0, \\ &p_t + \frac{\gamma p v_0}{v} u_h = \frac{(\gamma - 1)q}{v} Q, \end{aligned} \tag{2.6}$$

which may be written in the form

$$\partial_t \mathbf{U} + \mathbf{A}\partial_h \mathbf{U} = \mathbf{b}, \tag{2.7}$$

where

$$\mathbf{U} := \begin{pmatrix} v \\ u \\ p \end{pmatrix}, \quad \mathbf{A} := \begin{pmatrix} 0 & -v_0 & 0 \\ 0 & 0 & v_0 \\ 0 & \gamma p v_0/v & 0 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} 0 \\ 0 \\ (\gamma - 1)qQ/v \end{pmatrix}. \tag{2.8}$$

It may be easily verified that this system is hyperbolic and the characteristic equation is

$$\det(\mathbf{A} - \Lambda \mathbf{I}) = \Lambda\left(\Lambda^2 - \frac{\gamma p}{v} v_0^2\right) = 0. \tag{2.9}$$

The system (2.7) may be also written in conservative form,

$$\partial_t \mathbf{V} + \partial_h \mathbf{F} = \mathbf{0}, \tag{2.10}$$

where

$$\mathbf{V}^T := (v, u, e + \tfrac{1}{2} u^2), \quad \mathbf{F}^T := (v_0 u, v_0 p, v_0 u p). \tag{2.11}$$

This form is more suitable in order to study shock-wave propagation. Of course, to the system (2.7) we must associate the relation (2.5) which characterizes the time evolution of the progress variable λ.

The jump conditions which must hold across the shock front propagating into the reactant are given by the well-known Rankine-Hugoniot relations obtained from (2.5) and (2.10). By making the strong-shock assumption, the shock speed is given by

$$D = \frac{\gamma + 1}{2} u_1. \tag{2.12}$$

Moreover, taking into account that the precursor shock is purely fluid-dynamical, we obtain

$$\frac{v_1}{v_0} = \frac{\gamma + 1}{\gamma + 2}, \quad p_1 = \frac{(\gamma + 1)}{2 v_0} u_1^2, \tag{2.13}$$

where the subscripts 0 and 1 refer, respectively, to quantities evaluated ahead and behind the shock front. Because (2.12) and (2.13) are treated here like boundary conditions, we complete them by adding the following condition for λ:

$$\lambda_1 = 0. \tag{2.14}$$

In order to find invariant solutions of (2.7) and (2.5), we require the system to be invariant with respect to the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon T, \quad h^* = h + \varepsilon H, \quad u^* = u + \varepsilon U, \\ p^* &= p + \varepsilon P, \quad v^* = v + \varepsilon V, \quad \lambda^* = \lambda + \varepsilon L, \end{aligned} \tag{2.15}$$

where T, H, U, P, V, L are functions of t, h, u, p, v and λ, to be determined in order to characterize the possible groups of invariance. Of course, we also require the boundary conditions (2.12), (2.13), (2.14) to be invariant with respect to (2.15).

By using the results obtained in [3] we see that

$$T = at + d, \quad H = bh + d_1, \quad U = (b - a)u,$$
$$P = 2(b - a)p, \quad V = 0, \quad L = 2(b - a)\lambda, \tag{2.16}$$

where a, b, d, d_1 are constants, while the reaction rate must be a solution of the differential equation

$$pQ_p + \lambda Q_\lambda = \beta Q, \tag{2.17}$$

where

$$\beta = \frac{2b - 3a}{2b - 2a}. \tag{2.18}$$

Consequently, we get the following functional form for Q:

$$Q = K\left(\frac{p}{p_i}\right)^\beta F\left(\frac{v}{v_i}, \frac{p_i\lambda}{p}\right), \tag{2.19}$$

where K is a constant. F is an arbitrary function, and p_i, v_i are the initial values of pressure and volume of mixture. They are given from the jump conditions by

$$v_i = \frac{\gamma - 1}{\gamma + 2}v_0, \quad p_i = \frac{\gamma - 1}{2v_0}u_i^2, \tag{2.20}$$

where u_i denotes the initial particle velocity behind the shock.

In order to determine the similarity variable σ and the similarity solutions, we use the invariant-surface condition that allows us to obtain

$$\sigma = \frac{c_4 h + 1}{(c_3 t + 1)^{c_2}} \tag{2.21}$$

and

$$u(t, h) = (c_3 t + 1)^{c_2 - 1}u_i\hat{u}(\sigma),$$
$$p(t, h) = (c_3 t + 1)^{2(c_2 - 1)}p_i\hat{p}(\sigma),$$
$$v(t, h) = v_i\hat{v}(\sigma), \tag{2.22}$$
$$\lambda(t, h) = (c_3 t + 1)^{2(c_2 - 1)}\hat{\lambda}(\sigma),$$

where

$$c_2 = \frac{b}{a}, \quad c_3 = \frac{a}{d}, \quad c_4 = \frac{b}{d_1}, \tag{2.23}$$

and $\hat{u}$, $\hat{p}$, $\hat{v}$, $\hat{\lambda}$ are functions of σ.

3. Associated group and particular solutions

By substitution of (2.22) and (2.21) in (2.6) and (2.5), taking into account (2.19), recalling that the initial shock velocity is

$$D_0 = \frac{\gamma + 1}{2} u_i \tag{3.1}$$

and taking the similarity variable as independent variable, we get the following system of ordinary differential equations for $\hat{v}$, $\hat{u}$, $\hat{p}$:

$$\begin{aligned} c_2\sigma\hat{v}' + \frac{2}{\gamma - 1} D_0 \frac{c_4}{c_3} \hat{u}' &= 0, \\ c_2\sigma\hat{u}' + \frac{D_0 c_4}{c_3} \hat{p}' &= (c - 1)\hat{u}, \\ -\frac{2\gamma D_0}{\gamma - 1} \frac{c_4}{c_3} \frac{\hat{p}}{\hat{v}} \hat{u}' + c_2\sigma\hat{p}' &= 2(c_2 - 1)\hat{p} - \frac{2Kq\hat{p}^\beta F}{v_0 u_i^2 c_3}, \end{aligned} \tag{3.2}$$

and the ordinary differential equation for $\hat{\lambda}$,

$$c_3\sigma\hat{\lambda}' = 2(c_2 - 1)\hat{\lambda} - \frac{K\hat{p}^\beta F}{c_3}, \tag{3.3}$$

where, by (2.18) and (2.23),

$$c_2 = \frac{2\beta - 3}{2\beta - 2}, \tag{3.4}$$

and the initial conditions at $\sigma = 1$ are given by

$$\hat{u}(1) = \hat{v}(1) = \hat{p}(1) = 1, \quad \hat{\lambda}(1) = 0. \tag{3.5}$$

We can show that system (3.2) and equation (3.3) are invariant with respect to the following particular group:

$$\sigma^* = \omega\sigma, \quad \hat{v}^* = \omega^{c-2}\hat{v}, \quad \hat{u}^* = \omega^{c-1}\hat{u}, \quad \hat{p}^* = \omega^c\hat{p}, \quad \hat{\lambda}^* = \omega^c\hat{\lambda},$$

$$\omega \in \mathbb{R} - \{0\}, \quad c \in \mathbb{R} - \{2\}, \tag{3.7}$$

provided that F has the form

$$F = v^{c(1-\beta)/(c-2)} \mathscr{F}\left(\frac{\hat{\lambda}}{\hat{p}}\right). \tag{3.8}$$

We must observe that the invariance condition requires that the following condition must be satisfied

$$(c - 2)vF_v = c(1 - \beta)F. \tag{3.9}$$

If $c = 2$, it follows by (3.9) that $\beta = 1$, vice versa, if $\beta = 1$, it follows that $c = 2$ or $F = \mathscr{F}(\hat{\lambda}/\hat{p})$. Following [6], we call (3.7) the *associated stretching group* of the group generated by (2.19).

By introducing the new field variables

$$z = \frac{\hat{v}}{\sigma^{c-2}}, \quad w = \frac{\hat{u}}{\sigma^{c-1}}, \quad r = \frac{\hat{p}}{\sigma^{c}}, \quad y = \frac{\hat{\lambda}}{\sigma^{c}}, \tag{3.10}$$

and taking into account that

$$\begin{aligned} \hat{v}' &= z'\sigma^{c-2} + (c-2)\sigma^{c-3}z, \quad \hat{u}' = w'\sigma^{c-1} + (c-1)\sigma^{c-2}w, \\ \hat{p}' &= r'\sigma^{c} + c\sigma^{c-1}r, \quad \hat{\lambda}' = y'\sigma^{c} + c\sigma^{c-1}y, \end{aligned} \tag{3.11}$$

it follows from (3.2) and (3.3):

$$\begin{aligned} &\sigma c_2 z' + \frac{2D_0\sigma c_4}{(\gamma-1)c_3} w' = c_2(2-c)z + \frac{2D_0c_4}{(\gamma-1)c_3}(1-c)w, \\ &\sigma c_2 w' - \frac{\sigma D_0 c_4}{c_3} r' = c_2(1-c)z + \frac{D_0c_4}{c_3} cr + (c_2 - 1)w, \\ &-\frac{2\gamma D_0 c_4}{(\gamma-1)c_3}\sigma \frac{r}{z} w' + c_2\sigma r' = \frac{2\gamma D_0c_4}{(\gamma-1)c_3}(c-1)\frac{r}{z}w + \{c_2(2-c) - 2\}r \\ &\quad + \frac{2Kqr^{\beta}z^{c(1-\beta)/(c-2)}}{v_0u_i^2c_3}\mathscr{F}\left(\frac{y}{r}\right), \end{aligned} \tag{3.12}$$

and

$$c_2\sigma y' = \{c_2(2-c) - 2\}y - \frac{Kr^{\beta}z^{c(1-\beta)/(c-2)}}{c_3}\mathscr{F}\left(\frac{y}{r}\right). \tag{3.13}$$

Finally we observe that the system (3.12) can be written in the following matrix form:

$$\sigma(\tilde{\mathbf{A}} + c_2\mathbf{I})\frac{\mathrm{d}\tilde{\mathbf{u}}}{\mathrm{d}\sigma} = \tilde{\mathbf{B}}, \tag{3.14}$$

where

$$\tilde{\mathbf{A}} := \begin{pmatrix} 0 & \dfrac{2D_0c_4}{(\gamma - 1)c_3} & 0 \\ 0 & 0 & -\dfrac{D_0c_4}{c_3} \\ 0 & -\dfrac{2\gamma D_0 c_4 r}{(\gamma - 1)c_3 z} & 0 \end{pmatrix}, \quad \tilde{\mathbf{u}} := \begin{pmatrix} z \\ w \\ r \end{pmatrix},$$

$$\tilde{\mathbf{B}} := \begin{pmatrix} c_2(2 - c)z + \dfrac{2D_0c_4}{(\gamma - 1)c_3}(1 - c)w \\ \{c_2(2 - c)\}w + \dfrac{D_0c_4}{c_3}cr \\ \dfrac{2\gamma D_0c_4}{(\gamma - 1)c_3}(c - 1)\dfrac{r}{z}w + \{c_2(2 - c) - 2\}r - \dfrac{2Kqr^{\beta}z^{c(1-\beta)/(c-2)}\mathcal{F}\left(\dfrac{y}{r}\right)}{v_0u_i^2c_3} \end{pmatrix}$$

The system (3.12) can be also written in the following normal form,

$$\sigma\Delta\frac{\mathrm{d}z}{\mathrm{d}\sigma} = \Delta_1, \quad \sigma\Delta\frac{\mathrm{d}w}{\mathrm{d}\sigma} = \Delta_2, \quad \sigma\Delta\frac{\mathrm{d}r}{\mathrm{d}\sigma} = \Delta_3, \tag{3.16}$$

where

$$\Delta := \det(\tilde{\mathbf{A}} + c_2\mathbf{I}) = c_2^2 - \frac{2\gamma}{\gamma - 1}\frac{D_0^2c_4^2}{c_3^2}\frac{r}{z}, \tag{3.17}$$

$$\Delta_1 := B_1\Delta - \frac{2D_0c_4}{(\gamma - 1)c_3}\left(B_3c_2 + B_4\frac{c_4}{c_3}\right),$$

$$\Delta_2 := c_2\left(B_3c_2 + \frac{B_4c_4}{c_3}\right), \tag{3.18}$$

$$\Delta_3 := c_2\left(c_2B_3 + B_2\frac{2\gamma D_0c_4}{(\gamma - 1)c_3}\frac{r}{z}\right).$$

It is interesting to see that the system (3.12) and equation (3.13) may be reduced to autonomous form simply by choosing as new independent variable

$$\tau = \ln\sigma. \tag{3.19}$$

Therefore, we obtain

$$
\begin{aligned}
c_2 z' + \frac{2D_0 c_4}{c_3(\gamma - 1)} w' &= c_2(2 - c)z + \frac{2D_0 c_4}{(\gamma - 1)c_3}(1 - c)w, \\
c_2 w' - \frac{D_0 c_4}{c_3} r' &= \{c_2(2 - c) - 1\}w + \frac{D_0 c_4}{c_3} cr, \\
-\frac{2\gamma D_0 c_4}{(\gamma - 1)c_3}\frac{r}{z} w' + c_2 r' &= \frac{2\gamma D_0 c_4}{(\gamma - 1)c_3}(c - 1)\frac{r}{z} w \\
&\quad + \{c_2(2 - c) - 2\}r - \frac{2Kqr^\beta z^{c(1-\beta)/c-2)}}{v_0 u_i^2 c_3}\mathscr{F}\left(\frac{y}{r}\right), \\
c_2 y' &= \{c_2(2 - c) - 2\}y - \frac{Kr^\beta z^{c(1-\beta)/(c-2)}}{c_3}\mathscr{F}\left(\frac{y}{r}\right),
\end{aligned}
\tag{3.20}
$$

while the initial conditions become

$$
z(0) = 1, \quad r(0) = 1, \quad w(0) = 1, \quad y(0) = 0. \tag{3.21}
$$

Now basing ourselves on the procedure used in [8], we look for a particular solution of the Cauchy problem characterized by (3.20) and (3.21) in the form

$$
z = w = r = 1, \quad y = y(\tau). \tag{3.22}
$$

This implies that

$$
\begin{aligned}
c_2(2 - c) + \frac{2D_0 c_4}{(\gamma - 1)c_3}(1 - c) &= 0, \\
c_2(2 - c) + \frac{D_0 c_4}{c_3} c &= 1, \\
\frac{2\gamma D_0 c_4(c - 1)}{(\gamma - 1)c_3} + c_2(2 - c) - \frac{2q\tilde{\mathscr{F}}}{v_0 u_i^2 c_3} &= 2, \\
c_2 y' &= \{c_2(2 - c) - 2\}y - \frac{K}{c_3}\tilde{\mathscr{F}},
\end{aligned}
\tag{3.23}
$$

where

$$
\tilde{\mathscr{F}} := \mathscr{F}\left(\frac{y}{r}\right) = \text{const}, \tag{3.24}
$$

as follows from (3.20.III), taking into account (3.21).

From (3.23.IV) we obtain:

$$y = \frac{c_2 K\mathscr{F}}{c_3\{c_2(c-2)-2\}}\left[1-\exp\left\{\frac{c_2(c-2)-2}{c_2}\tau\right\}\right]. \tag{3.25}$$

The other three equations (3.23) allow us to determine the values of c, c_3, c_4. In fact, the remaining quantities in the problem are assumed to be known either from the form of the rate law (K, β and $\mathscr{F}$), the physical characteristic of the reactive material (γ and q), or the initial piston energy (u_i).

By assuming the constant c_2 in suitable form,

$$c_2 = \frac{4}{3\gamma-7}\quad\left(\beta = \frac{29-9\gamma}{2(11-3\gamma)}\right), \tag{3.26}$$

it follows from (3.23, I, II, III) that:

$$c = 3,\quad c_3 = -\frac{(3\gamma-7)q\tilde{\mathscr{F}}}{(5\gamma-9)v_0u_i^2},\quad c_4 = \frac{\gamma-1)q\tilde{\mathscr{F}}}{(5\gamma-9)D_0v_0u_i^2}, \tag{3.27}$$

while (3.25) becomes

$$y = \frac{2v_0u_i^2(5\gamma-9)K}{3(3-2\gamma)(3\gamma-7)q}\left[1-\exp\left\{\frac{3(3-\gamma)}{2}\tau\right\}\right]. \tag{3.28}$$

As expected, the above values are the only possible ones for c, c_3 and c_4.

Finally, we get a particular similarity solution of the form

$$u(t,h) = \left(-\frac{(3\gamma-7)q\tilde{\mathscr{F}}}{(5\gamma-g)v_0u_i^2}t+1\right)^{-(3\gamma-11)/(3\gamma-7)} u_i\sigma^2,$$

$$p(t,h) = \left(-\frac{(3\gamma-7)q\tilde{\mathscr{F}}}{(5\gamma-9)v_0u_i^2}t+1\right)^{-2(3\gamma-11)/(3\gamma-7)} p_i\sigma^3,$$

$$v(t,h) = v_i\sigma,$$

$$\lambda(t,h) = \frac{2v_0u_i^2K(5\gamma-9)}{3(3-2\gamma)(3\gamma-7)q}\left(-\frac{(3\gamma-7)q\tilde{\mathscr{F}}}{(5\gamma-9)v_0u_i^2}t+1\right)^{-2(3\gamma-11)/(3\gamma-7)} \sigma^3\left(1-\sigma^{3(3-\gamma)/2}\right),$$

where

$$\sigma = \frac{\{(\gamma-1)q\tilde{\mathscr{F}}/(5\gamma-9)D_0v_0u_i^2\}\,h+1}{[-\{(3\gamma-7)q\tilde{\mathscr{F}}/(5\gamma-9)v_0u_i^2\}\,t+1]^{4/(3\gamma-7)}}.$$

4. Weak discontinuities

We consider the case when a similarity solution suffers a jump in the first-order derivatives of $\tilde{\mathbf{u}}$ across the similarity curve characterized by the value σ_f:

$$\sigma_f(c_3 t + 1)^{c_2} - c_4 h = 1. \tag{4.1}$$

This value σ_f is determined as a root of the characteristic polynomial associated with the hyperbolic system (2.7). In fact, taking into account (2.21), we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \Lambda = \frac{c_2 c_3}{c_4}(c_3 t + 1)^{c_2 - 1}\sigma, \tag{4.2}$$

for which we obtain from (2.9):

$$\det(\mathbf{A} - \Lambda\mathbf{I}) = \sigma^{-3}\Delta. \tag{4.3}$$

This relation confirms that the singularities will occur across the characteristic curves.
Because singularities will appear when

$$\Delta(z(\sigma), w(\sigma), r(\sigma), y(\sigma)) = 0, \tag{4.4}$$

then, if we know $z(\sigma)$, $w(\sigma)$, $r(\sigma)$, $y(\sigma)$, condition (4.4) determines the possible values of σ_f.
In order that the solutions determined by (4.4) for values of σ close to σ_f be continued, we require that Δ_1, Δ_2 and Δ_3 are also zero for $\sigma = \sigma_f$. Nevertheless, we can show that the conditions

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0 \tag{4.5}$$

are not independent [7], so that we may consider the following system:

$$\Delta = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0. \tag{4.6}$$

Now fixing, for instance, in order to simplify the calculations:

$$c = 1, \tag{4.6'}$$

and assuming

$$\beta = 2, \quad (c_2 = \tfrac{1}{2}), \quad \mathscr{F}\left(\frac{y}{r}\right) = 1 - \frac{y}{r}, \tag{4.7}$$

we can write the vector $\tilde{\mathbf{B}}$ in the following form:

$$\tilde{\mathbf{B}} := \begin{pmatrix} \dfrac{z}{2} \\ \dfrac{D_0 c_4 r}{c_3} - \dfrac{w}{2} \\ \frac{3}{2}r - \dfrac{2Kqrz}{v_0 u_i^2 c_3}(r - y) \end{pmatrix}. \tag{4.8}$$

Therefore, from (4.6), and taking into account (3.17), (3.18) and (4.8), we obtain

$$\begin{aligned} z_f &= \frac{4\gamma}{\gamma - 1}\frac{D^2 c_4^2}{c^2}\chi(y_f)^*, \\ w_f &= \frac{Dc_4}{c_3}\chi(y_f), \\ r_f &= \tfrac{1}{2}\chi(y_f) \end{aligned} \tag{4.9}$$

where

$$\chi(y_f) := y_f \pm \sqrt{y_f^2 - 3E} > 0 \tag{4.10}$$

with

$$y_f^2 - 3E \geqslant 0 \tag{4.11}$$

and

$$E = \frac{(\gamma - 1)v_0 u_i^2 c_3^2}{8KqD_0 c_4^2}. \tag{4.12}$$

Now we guess a value of $\bar{\sigma}_f$ and a value $\bar{y}_f$ such that $\bar{y}_f^2 \geqslant 3E$ and by using (4.9), we calculate, $\bar{z}(\bar{y}_f)$, $\bar{w}(\bar{y}_f)$, $r(\bar{y}_f)$. These values $\bar{z}_f$, $\bar{w}_f$, $\bar{r}_f$, $\bar{y}_f$ can be used to integrate the system (3.12) and the equation (3.13), with assumptions (4.6′), from $\bar{\sigma}_f$ inward to $\sigma = 1$, yielding $\bar{z}(1)$, $\bar{w}(1)$, $\bar{r}(1)$, $\bar{y}(1)$.

Then, taking into account that from (3.5) and (3.10), with (4.6′) and (4.7), we have

$$z(1) = w(1) = r(1) = 1, \quad y(1) = 0, \tag{4.13}$$

by using the invariance properties (3.7) and (3.10), we determine ω from

$$\bar{w}(1) = \omega w(1), \tag{4.13'}$$

* With $(\ )_f$ we denote a function evaluated for $\sigma = \sigma_f$.

so that the effective values of σ_f and y_f will be given by

$$\sigma_f = \frac{\bar{\sigma}_f}{\omega}, \quad y_f = \frac{\bar{y}_f}{\omega^2}. \tag{4.14}$$

As is known [7],

$$(\tilde{\mathbf{A}}_f + c_2\mathbf{I})\boldsymbol{\pi} = 0, \tag{4.15}$$

so that

$$\boldsymbol{\pi} = \pi_f \mathbf{d}_f, \tag{4.16}$$

where $\mathbf{d}_f$ is the right eigenvector of $\tilde{\mathbf{A}}_f$ corresponding to the eigenvalue $\Lambda(\sigma_f)$. Therefore, after some classical developments, we get

$$((\nabla_{\bar{\mathbf{u}}}\Lambda \cdot \mathbf{d})(\mathbf{1} \cdot \mathbf{d}))_f \sigma_f \pi_f = (\nabla_{\bar{\mathbf{u}}}(\mathbf{1} \cdot \mathbf{B})\mathbf{d})_f, \tag{4.17}$$

from which, taking into account that

$$\Lambda = v \frac{D_0 c_4}{c_3}, \quad v := \pm \left(\frac{2\gamma}{\gamma - 1} \frac{r}{z}\right)^{1/2}, \tag{4.18}$$

and that the corresponding left and right eigenvectors are

$$\mathbf{1} = (0, v, -1), \quad \mathbf{d} = \left(\frac{2}{\gamma - 1}, v, -v^2\right), \tag{4.19}$$

it follows that

$$\pi_f = \frac{4\gamma D_0^2 c_4^2 \chi_f}{(\gamma^2 - 1)v_0 c_3^5 \sigma_f} (c_3^3(\gamma - 1)v_0 + c_4^2 K(\gamma + 1)\gamma q \chi_f^2). \tag{4.20}$$

Then the evolution law of discontinuities will be [9]

$$\boldsymbol{\pi} = \pi_f \begin{bmatrix} \dfrac{2v_i \sigma_f}{\gamma - 1} \\ (c_3 t + 1)^{1/2} u_i v_f \\ -(c_3 t + 1) p_i v_f^2 \end{bmatrix}. \tag{4.21}$$

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